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Stability analysis of Holographic Dark Energy Model with various Interaction terms

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Abstract The paper deals with an interacting two fluid cosmological model in the background of homogeneous and isotropic flat FLRW space-time geometry. Among the two fluids one of them is considered as cold dark matter while the other fluid is chosen as holographic/ modified holographic dark energy having event horizon/ IR cut off. Due to coupled and complicated field equations dynamical system analysis has been used. The equilibrium points have been analyzed from cosmological perspective and stability of the equilibrium points has been discussed.

Introduction:

A number of cosmological observations, including those from Type Ia Supernova, Cosmic Microwave Background Radiation (CMBR) anisotropies [1] , Baryon Acoustic Oscillation [2], X-ray experiments, and Large Scale Structures (LSS) [3], have clearly shown that our Universe is going through an accelerated phase of expansion [4,5] . Dark energy, an unidentified exotic matter with high negative pressure, is introduced into mainstream cosmology to account for this observed reality. Nonetheless, the nature of dark energy remains a mystery, making it one of the main problems facing cosmology today. With an equation of state $\omega_d = \frac{p_d}{\rho_d} = -1$ (where p_d and ρ_d are the thermodynamic pressure and energy density of the dark energy, respectively), the cosmological constant Λ represents a vacuum energy density and is the most basic description of dark energy. In order to fairly explain the empirical evidence (the accelerated expansion of the universe today), the earliest and most basic theoretical contender (ΛCDM) was proposed. There are two major issues with Λ, which are the cosmic coincidence problems and the fine tuning issues. The theoretical value of Λ is several orders of magnitude larger than the current observational value in the former problem (roughly 10^{123} higher than what we observe [6]). In the latter problem, the mystery of the current densities of dark energy (DE) and dark matter (DM) of the same order, albeit evolving in different ways, is addressed. Several dynamic dark energy models with distinct equations of state during the universe's expansion have been proposed to address these issues [7,8] . Holographic Dark Energy (HDE) models have received a lot of interest among these [9]. The holographic principle of quantum gravity, which asserts that a system's entropy scales with its surface area L^2 rather than its volume, is the foundation upon which HDE models are built. According to this principle, the zero point energy of the system with size L can be bounded by the mass of a black hole with the same size [10] as follows

$$\rho_{\Lambda} \leq L^{-2} M_p^2, \quad (1)$$

where ρ_{Λ} is t he quantum vacuum energy density and $M_p = \frac{1}{\sqrt{8\pi G}}$ is the plank mass.

An analogy between the infra-red (*IR*) cut-off encoded in the scale L and the ultraviolet (UV) cut-off, defined by ρ_{Λ}, is described by this inequality. In cosmology, the dark energy density of the universe, ρ_d, can be thought of as equal to the vacuum energy, or $\rho_d = \rho_{\Lambda}$. Saturating the inequality yields the biggest *IR*-cutoff L from effective quantum field theory. As a result, the dark energy density (the vacuum energy density) can be written as

$$\rho_d = \frac{3M_p^2 C^2}{L^2}, \quad (2)$$

where ' C ' is a dimensionless numerical parameter which is estimated by observational data: for flat Universe (i.e. for k=0) it is obtained that $C = 0.1818^{+0.173}_{-0.097}$ and in the case of non-flat Universe (i.e. for $k = \pm 1$) it is obtained that $C = 0.815^{+0.179}_{-0.139}$ (reference [11], [12]).

The Hubble radius [9,13], the Future event horizon [10,14], and Ricci's scalar curvature [15,16] are the three most commonly utilized options for the *IR*-cutoff in the literature. Although it provides the right energy density, the Hubble radius $L = H^{-1}$ (H is the standard Hubble parameter) is unable to provide the correct equation of state for dark energy. This choice of *IR*-cutoff is proposed to explain both of the cosmological constant difficulties for the future event horizon $L = R_E$. However, Granda and Oliveros [17] introduced the Ricci scalar curvature, $L = (\dot{H} + 2H^2)^{-\frac{1}{2}}$, which is based on the spacetime scalar curvature as *IR*-cutoff and has a similarity to the size of maximal perturbation that causes a black hole to form. This *IR*-cutoff is also reasonably excellent at fitting the observational data and may remove the fine tuning and coincidence issues.
A system governed by gravity must have a different number of relevant degrees of freedom in proportion to the extent of the surface that surrounds it. This is the holographic principle [18]. Accordingly, cosmologists have hypothesized that the use of the holographic principle could provide some clues on the enigmatic and unknowable nature of DE. Holographic dark energy (HDE) models are any dynamical DE models that are based on this idea. From the point of view of quantum gravity the energy density of any given region should be bounded that ascribed to a Schwarzschild blackhole (within the volumn), i.e.,

$$\rho_{\Lambda} \leq L^{-2} M_p^2, \quad (3)$$

where ρ_{Λ} is the quantum vacuum energy density and $M_p = \frac{1}{\sqrt{8\pi G}}$ is the Planck mass .
In contrast to the equivalent non-interacting scenarios, the cosmic coincidence problem can also be resolved by the holographic dark energy models that interact with dark matter (see sections 1 and 2) [19,20]. Furthermore, it has been previously investigated that for certain interaction models, the equation of state for the holographic dark energy may cross the phantom divide line if the future event horizon is set to the infra-red (IR)-cutoff [21]. This situation likewise holds true when the universe is curved [22]. As far as we are aware from the literature, non-interacting holographic dark energy models are incapable of crossing the phantom split line. We consult Ref. [23] for a comparison of interacting and non-interacting holographic dark energy models. As we discussed in section 2, there are a variety of options for the *IR*-cutoff, but three are frequently employed in the literature: the Hubble radius, the future event horizon, and the Ricci length scale. The scenarios that correlate to various *IR*-cut-off lengths will be briefly explained.
The Holographic Dark Energy model (HDE) with an event horizon of *IR*-cutoff is the first thing we look at in this chapter. The modified holographic dark energy model at Ricci's Scale (MHRDE) is next examined. This chapter is structured as follows: first, we go over the model's fundamental equations and interaction terms. Next, we analyze the dynamical system of the interacting HDE model by applying the field equations to autonomous systems. Finally, we look at the interacting MHRDE model. In order to study non-hyperbolic equilibrium points and determine potential bifurcation values, we develop Lyapunov functions in this section. The discussion of the dynamical system analysis's cosmic implications concludes.

1. Primary Equations and construction of an autonomous system

In large scale universe can be treated as homogeneous and isotropic flat FRW. Few models have been proposed to correctly describe that universe is filled up with dark matter (ρ_{dm}) in the form of dust and HDE in perfect fluid which satisfies the state equation $p_{de} = w_{de}\rho_{de}$ where p_{de} is the thermodynamic pressure and ρ_{de} is the energy density. There is no gravitational forces between DM and DE.
The Einstein field equations are

$$3M_p^2 H^2 = \rho_{dm} + \rho_{de} \quad (4)$$

where M_p^2 is the reduced Plank mass, $M_p^2 = \frac{1}{8\pi G}$

$$2\dot{H} = -\rho_{dm} - (1 + w_{de})\rho_{de}. \tag{5}$$

Applying equations (4) and (5), we get acceleration of universe

$$\ddot{a} = a(\dot{H} + H^2) = -\frac{a}{6}[\rho_{dm} + \rho_{de}(1 + 3w_{de})]. \tag{6}$$

Now for cosmic acceleration we need $w_{de} < -\frac{1}{3}$. Introducing new parameters in the equation (5) we have

$$\Omega_{dm} + \Omega_{de} = 1 \tag{7}$$

where $\Omega_{dm} = \frac{\rho_{dm}}{3M_p^2H^2} = \frac{u}{1+u}$, $\Omega_{de} = \frac{\rho_{de}}{3M_p^2H^2} = \frac{1}{1+u}$ and $u = \frac{\rho_{dm}}{\rho_{de}}$, the energy densities ratio. Now introducing interaction term between dark energy and dark matter, the energy conservation equations turns into the form

$$\dot{\rho_{dm}} + 3H\rho_{dm} = Q, \tag{8}$$

and

$$\dot{\rho_{de}} + 3H(1 + w_{de})\rho_{de} = -Q. \tag{9}$$

The interaction term Q have different form. In this article we assume $Q > 0$ i.e dark energy converted to dark matter. $Q > 0$ mainly confirm three postulate namely it satisfies (i) Le Chateliers principle, (ii) validity of the second law of thermodynamics, (iii) and in favour of resolving the coincidence problem.

We only discuss about dark energy and dark matter, so we shall not consider baryonic matter in the interaction term. In this paper consider five different interaction term separately such as

(I) $Q = 3b^2H\sqrt{\rho_{de}\rho_{dm}}$,

(II) $Q = 3b^2H\left(\frac{\rho_{de}\rho_{dm}}{\rho_{dm} + \rho_{de}}\right)$,

(III) $Q = 3Hb^2\rho_{dm}$,

(IV) $Q = 3b^2\rho_{dm}$,

(V) $Q = \frac{3b^2\rho_{dm}\rho_{de}}{H}$.

2. HOLOGRAPHIC DARK ENERGY MODEL WITH EVENT HORIZON AS IR-CUTOFF

We choose the IR-cutoff as event horizon of the universe i.e radius R_E is defined by the improper integral:

$$R_E = a\int_t^\infty \frac{dt}{a} = a\int_a^\infty \frac{da}{Ha^2}. \tag{10}$$

It is clear that the above integral converges only when the strong energy condition is violated. Therefore in the present acclerating phase the improper integral always exists. Now assuming $L = R_E$ we have

$$\rho_{de} = \frac{3M_p^2C^2}{R_E^2}. \tag{11}$$

Now using the equations (10) and (11) and also expression for dark energy density ρ_{de} from (8), for any interaction term Q, the EoS takes the following form

$$w_{de} = -\frac{1}{3} - \frac{2\sqrt{\Omega_{de}}}{3c} - \frac{Q}{3H\rho_{de}}. \tag{12}$$

The second Friedman equation (4) and the dark energy conservation equation turn into an autonomous system (which is correspond to the phase plane (H,ρ_{de})) as

$$\begin{aligned}\dot{H} &= -\frac{3H^2}{2}\left[1 - \frac{\Omega_{de}}{3} - \frac{2\Omega_{de}^{\frac{2}{3}}}{3c} - \frac{Q}{3H\rho}\right],\\ \dot{\rho}_{de} &= 2H\rho_{de}\left[\frac{\sqrt{\Omega_{de}}}{c} - 1\right].\end{aligned} \tag{13}$$

Now this autonomous system has a line of equilibrium points for $\Omega_{de} = c^2$ in the phase plane (H,ρ_{de}).

We now analyze the phase space of the system with different choices of interaction terms separately. Let us first choose $Q = 3b^2 H\sqrt{\rho_{de}\rho_{dm}}$, the above autonomous systems reduces to

$$\begin{aligned} \dot{H} &= -\frac{1}{2}\left[3H^2 - \frac{\rho_{de}}{3} - \frac{2}{\sqrt{3}cH}\rho_{de}^{\frac{3}{2}} - b^2\sqrt{3H^2\rho_{de} - \rho_{de}^2}\right], \\ \dot{\rho}_{de} &= 2\rho_{de}H\left[\frac{\sqrt{\rho_{de}}}{\sqrt{3}Hc} - 1\right]. \end{aligned} \quad (14)$$

We choose $\rho_{de} = v^2$, the equations (14) takes the following form

$$\begin{aligned} \dot{H} &= -\frac{1}{2}\left[3H^2 - \frac{v^2}{3} - \frac{2}{\sqrt{3}cH}v^3 - b^2 v\sqrt{3H^2 - v^2}\right], \\ \dot{v} &= 2vH\left[\frac{v}{\sqrt{3}cH} - 1\right]. \end{aligned}$$

The only physically meaningful equilibrium point corresponding to the above autonomous system is $A = \left(H_e, \sqrt{3}cH_e\right)$, whenever $b^2 = \frac{(3-7c^2)}{c\sqrt{1-c^2}}$ and H_e is a non zero real number. The Jacobian matrix at A corresponding to the above autonomous system can be written as

$$J(A) = \begin{pmatrix} \dfrac{3(2c^4 + 7c^2 - 5)H_e}{2(1-c^2)} & \dfrac{(43c^4 - 28c^2 + 9)H_e}{2\sqrt{3}c(1-c^2)} \\ -2\sqrt{3}cH_e & 2H_e \end{pmatrix}$$

and the eigenvalues are $\lambda_{1,2} = \frac{\left(-6c^4 - 17c^2 + 11 \mp \sqrt{36c^8 + 988c^6 - 739c^4 - 358c^2 + 217}\right)H_e}{4(c^2-1)}$ for all $H_e \in R - \{0\}$. We can clearly see that the nature of the equilibrium point A explicitly depends on the value of c. The equilibrium point A is hyperbolic for $c \neq \pm\, 0.655$ and nonhyperbolic for $c = \pm\, 0.655$ (because $\lambda_1 = 0$ for $c = \pm\, 0.655$ and $\lambda_2 \neq 0$ for all c). From the eigenvalues, we can conclude that for $H_e > 0$, the eigenvalues are complex conjugate to each other with negative real part while $c \in (0.667, 0.737) \cup (-0.737, -0.667)$ and the real part is positive while $c \in (-0.918, -0.737) \cup (0.737, 0.918)$. Further, for $H_e < 0$, the eigenvalues are complex conjugate to each other with negative real part while $c \in (-0.918, -0.737) \cup (0.737, 0.918)$ and the real part is positive while $c \in (0.667, 0.737) \cup (-0.737, -0.667)$. The real parts of λ_1 and λ_2 are zero when $c = \pm\, 0.737$. We can also see that for $-0.655 < c < 0.655$, $\lambda_1 > 0$ and $\lambda_2 < 0$ while $H_e > 0$; and $\lambda_1 < 0$ and $\lambda_2 > 0$ while $H_e < 0$. Hence, using Hartman-Grobman theorem, we can conclude that the equilibrium point A is a stable focus (spiral sink) if $c \in (0.667, 0.737) \cup (-0.737, -0.667)$ and $H_e > 0$ or $c \in (-0.918, -0.737) \cup (0.737, 0.918)$ and $H_e < 0$; the equilibrium point A is an unstable focus (spiral source) if $c \in (-0.918, -0.737) \cup (0.737, 0.918)$ and $H_e > 0$ or $c \in (0.667, 0.737) \cup (-0.737, -0.667)$ and $H_e < 0$; the equilibrium point A is a center if $c = \pm\, 0.737$; and for $-0.655 < c < 0.655$ the equilibrium point A is a saddle point and unstable in nature (see figure 1).

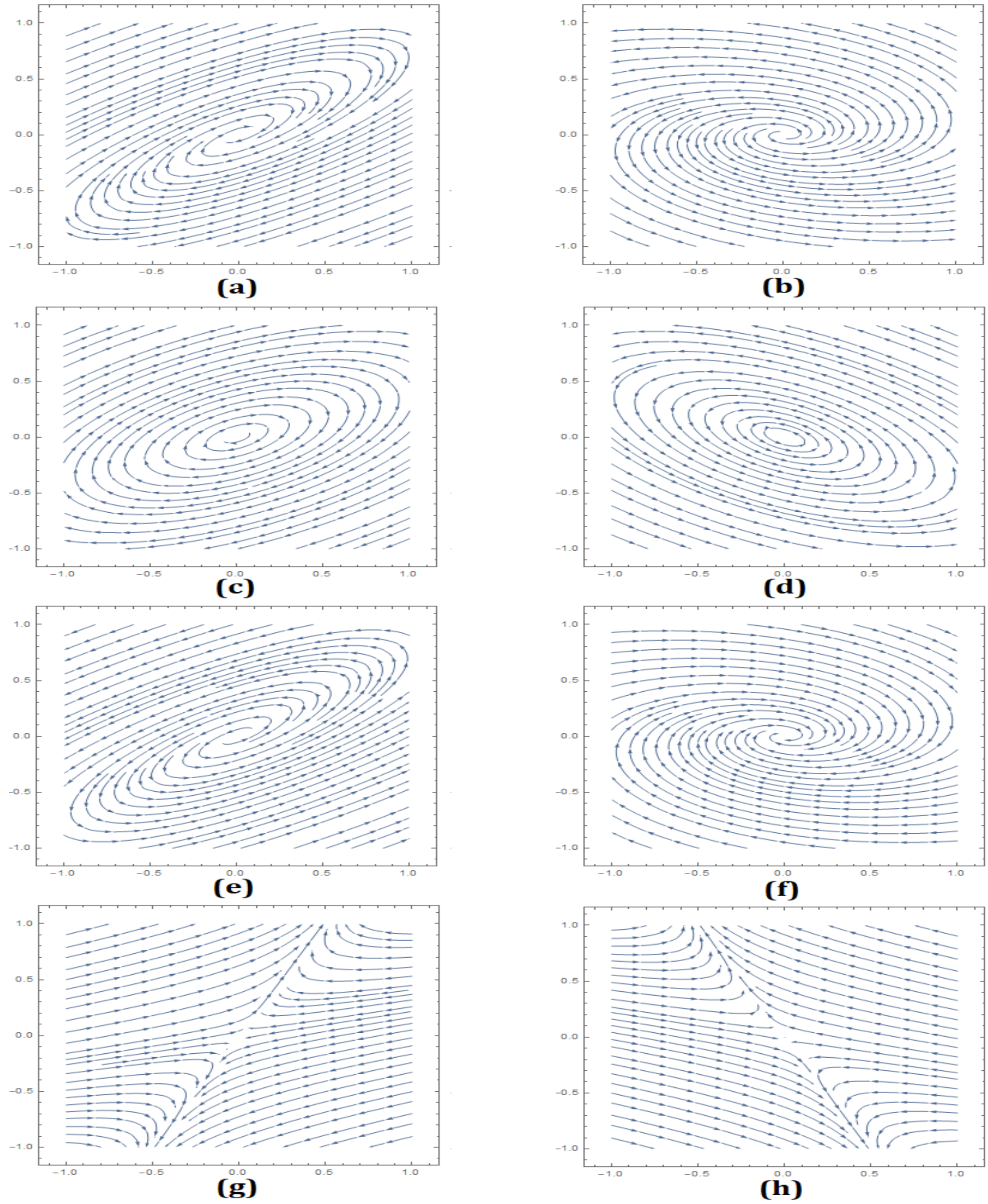

Figure 1. Profile of the global analysis in finite phase space for several values of c *and* H_e *corresponding to the equilibrium point* A*. The horizontal axis represents variable 'H' and the vertical axis represents variable 'v'. (a) corresponds to* $c = 0.7$ *and* $H_e = 1$ *and this phase portrait shows that the origin is a stable focus, (b) corresponds to* $c = -0.8$ *and* $H_e = 1$*: this phase portrait indicates that the origin is an unstable focus, (c) corresponds to* $c = 0.737$ *and* $H_e = 1$*: this phase portrait shows that the type of stability of the origin is center, (d) corresponds to* $c = -0.737$ *and* $H_e = 1$*: the type of stability of the origin in this case is also center (but the direction of vector field in anticlockwise direction), (e) corresponds to* $c = 0.7$ *and* $H_e = -1$*: the origin exhibits its unstable focus behavior in this case, (f) corresponds to* $c = -0.8$ *and* $H_e = -1$*: this phase portrait shows that the type of stability of the origin is a stable focus, (g) corresponds to* $c = 0.5$ *and* $H_e = 1$*: this phase portrait shows that the origin is a saddle node, (h) corresponds to* $c = -0.5$ *and* $H_e = 1$*: the type of stability of the origin in this case also saddle node and unstable in nature.*

Now the conservation equations can be expressed in the form

$$\begin{aligned} \dot{\rho}_{dm} &= \sqrt{3(\rho_{dm} + \rho_{de})}[b^2 \sqrt{\rho_{de}\rho_{dm}} - \rho_{dm}], \\ \dot{\rho}_{de} &= -\sqrt{3(\rho_{dm} + \rho_{de})}\left[b^2 \sqrt{\rho_{dm}\rho_{de}} - (1 + \omega_d)\rho_{de}\right]. \end{aligned} \quad (15)$$

The above system is $C^1(S)$ where S is the open set on R^2 defined by $S=[(\rho_{dm},\rho_{de})\in R^2|\rho_{dm}>0,\rho_{de}>0]$. The set of equations in (14) will represent an autonomous systems provided for the dark energy is constant. The equilibrium points for the autonomous systems $\dot{\rho}_{dm}=0$ and $\dot{\rho}_{de}=0$, we get either $\rho_{dm}=0$, $\rho_{de}=0$ or $\rho_{dm}=b^4\rho_{de}$ provided $\omega_d<-1$ which assists cosmic acceleration. All the equilibrium points are non-hyperbolic equilibrium points, we now construct suitable Liapunov function so that we can analyze the stability of the system for all equilibrium points. Let us construct $V(\rho_{dm},\rho_{de})=(\rho_{dm}-b^4\rho_{de})^2$, this function over R^2 continuously differentiable. For the equilibrium points (ρ_{dm},ρ_{de}) we have $V(\rho_{dm},\rho_{de})=0$ and $V(\rho_{dm},\rho_{de})>0$ for all other points.

$$\dot{V}(\rho_{dm},\rho_{de})=2(\rho_{dm}-b^4\rho_{de})(\dot{\rho}_{dm}-b^4\rho_{de})=-3H(\rho_{dm}-b^4\rho_{de})^2$$

Now from the above equation, we can notice that $\dot{V}(\rho_{dm},\rho_{de})<0$ for all point in S except equilibrium points. From Lyapunov stability analysis it is clear that the non-hyperbolic equilibrium points are asymptotically stable.

Next we consider the interaction term $Q=3Hb^2\left(\frac{\rho_{dm}\rho_{de}}{\rho_{dm}+\rho_{de}}\right)$, we analyze the evolution equations for the given interaction term. For the interaction term Q, the autonomous system take the form as

$$\begin{aligned}\dot{H} &= -\frac{1}{2}\left[3H^2-\frac{\rho_{de}}{3}-\frac{2\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}Hc}-\frac{b^2\rho_{de}}{3H^2}(3H^2-\rho_{de})\right],\\ \dot{\rho_{de}} &= 2\rho_{de}H\left[\frac{\sqrt{\Omega_{de}}}{c}-1\right].\end{aligned} \qquad (16)$$

Choosing $\rho_{de}=v^2$ and $\Omega_{de}=\frac{\rho_{de}}{3H^2}$, the equations of (16) converted to

$$\begin{aligned}\dot{H} &= -\frac{1}{2}\left[3H^2-\frac{v^2}{3}-\frac{2v^2}{3\sqrt{3}cH}-\frac{v^2b^2}{3H^2}(3H^2-v^2)\right],\\ \dot{v} &= Hv\left[\frac{v}{\sqrt{3}cH}-1\right].\end{aligned} \qquad (17)$$

This system is continuously differentiable on R^2 for $H\neq 0$. The only equilibrium point of this system is $B=\left(\frac{2c}{\sqrt{3}(3-c^2-3b^2c^2+3b^2c^4)},\frac{2c^2}{3-c^2-3b^2c^2+3b^2c^4}\right)$ for all $H\in R-\{0\}$. The Jacobian matrix corresponding to equations (17) is

$$J=\begin{pmatrix}-3H-\frac{v^3}{3\sqrt{3}cH^2}+\frac{b^2v^4}{3H^3} & \frac{v}{3}+\frac{2v}{3\sqrt{3}H}+b^2v-\frac{2b^2v^3}{3H^2}\\ -v & \frac{2v}{\sqrt{3}c}-H\end{pmatrix}.$$

The eigenvalues of the Jacobian matrix at the point B are

$$\lambda_{1,2}=\frac{c\left(3\sqrt{3}b^2c^4-\sqrt{3}c^2-2\sqrt{3}\mp\sqrt{27b^4c^8-18b^2c^6-36b^2c^5+36b^2c^3-36b^2c^2+3c^4+12c^3+12c^2-36c+48}\right)}{3\left(3b^2c^4-3b^2c^2-c^2+3\right)}.$$

For the hyperbolic case, using the Hartman-Grobman theorem, the stability features of the equilibrium point B in the regions of the parameter space (b,c) are shown in the figure 2. We can also note that the eigenvalues are purely imaginary if $b^2=\frac{2+c^2}{3c^4}$, that is, the type of stability of the equilibrium point is center if $b^2=\frac{2+c^2}{3c^4}$.

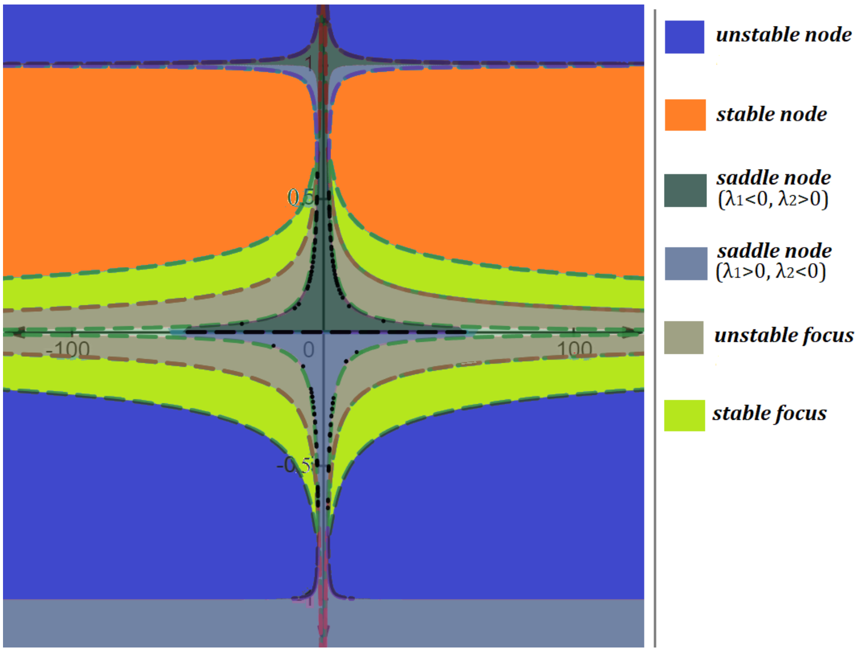

Figure 2. This figure shows the regions of the parameter space (b,c) corresponding to the stability features for the equilibrium point B. The horizontal axis represents parameter 'b' and the vertical axis represents parameter 'c'.

We further investigate and analyse the system for non hyperbolic instance, for this we write the Conservation equations in this format

$$\begin{aligned}\dot{\rho}_{dm} &= \sqrt{3}\rho_{dm}\left[\frac{b^2\rho_{de}}{\sqrt{\rho_{dm}+\rho_{de}}}-\sqrt{\rho_{dm}+\rho_{de}}\right],\\ \dot{\rho}_{de} &= -\sqrt{3}\rho_{de}\left[\frac{b^2\rho_{dm}}{\sqrt{\rho_{dm}+\rho_{de}}}-\frac{(1+\omega_{de})\sqrt{\rho_{de}+\rho_{dm}}}{3}\right].\end{aligned} \tag{18}$$

The above system is $C^1(E)$ where E is the open set on R^2 defined by $E=\{(\rho_{dm},\rho_{de})\in R^2|\rho_{dm}>0,\rho_{de}>0\}$. The set of equations in (18) will represent an autonomous systems provided for the dark energy is constant. The fixed points for the autonomous system $\dot{\rho}_{dm}=0$ and $\dot{\rho}_{de}=0$, we get either $\rho_{dm}=0$, $\rho_{de}=0$ or $\rho_{dm}=(b^4-1)\rho_{de}$, provided $\omega_d<-1$ which assists cosmic acceleration. All the fixed points are non-hyperbolic equilibrium points, we now construct suitable Lyapunov function so that we can analyze the stability of the system for all equilibrium points. Let us consider $V(\rho_{dm},\rho_{de})=(\rho_{dm}-(b^4-1)\rho_{de})$, this function continuously differentiable over R^2. For all equilibrium point (ρ_{dm},ρ_{de}) we have $V(\rho_{dm},\rho_{de})=0$ and $V(\rho_{dm},\rho_{de})>0$ for all other points.

$$\dot{V}(\rho_{dm},\rho_{de}) = 2(\rho_{dm}-(b^4-1)\rho_{de})(\dot{\rho}_{dm}-(b^4-1)\dot{\rho}_{de}) = -3H[\rho_{dm}-(b^4-1)\rho_{de}]^2.$$

Now from the above equation $\dot{V}(\rho_{dm},\rho_{de})<0$ for all points of E except equilibrium points. From this it is clear that all the equilibrium points are asymptotically stable.

Next we choose the interacting term Q as $Q=3Hb^2\rho_{dm}$, the second Friedman equation and the energy conservation equation for Dark Energy takes the form as,

$$\begin{aligned}\dot{H} &= -\frac{3H^2}{2}+\frac{\rho_{de}}{6}+\frac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH}+3H^2b^2-b^2\rho_{de},\\ \dot{\rho}_{de} &= \frac{2\rho_{de}^{\frac{3}{2}}}{\sqrt{3}c}-2H\rho_{de}.\end{aligned} \tag{19}$$

Equations (19) forms an autonomous system. We try to analyze the phase plane (H,ρ_{de}). The perturbation matrix of the above system is

$$J(H,\rho_{de}) = \begin{pmatrix} -3H + 6b^2H - \dfrac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH^2} & \dfrac{1}{6} - b^2 + \dfrac{\sqrt{\rho_{de}}}{2\sqrt{3}cH} \\ -2\rho_{de} & \dfrac{\sqrt{3\rho_{de}}}{c} - 2H \end{pmatrix}$$

The only physically meaningful equilibrium point corresponding to the above system is $C \equiv (H_e, 3c^2H_e^2)$, provided $b^2 = \frac{1}{2}\left(\frac{c^2-1}{3c^2-1}\right)$ for all $H_e \in R - \{0\}$. The eigenvalues of the Jacobian matrix at the point C are

$$\lambda_1 = -\frac{6c^2H_e}{3c^2-1}, \quad \lambda_2 = (1-c^2)H_e.$$

From the eigenvalues, we can say that C is hyperbolic if $c \neq 0, \pm 1$ and nonhyperbolic if $c \in \{-1,0,1\}$. Now we can observe that while $H_e > 0$ the eigenvalues $\lambda_1, \lambda_2 > 0$ if $c \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0, \frac{1}{\sqrt{3}}\right)$; $\lambda_1, \lambda_2 < 0$ for $|c| > 1$; and $\lambda_1 < 0, \lambda_2 > 0$ if $c \in \left(-1, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$. Further, notice that while $H_e < 0$ the eigenvalues $\lambda_1, \lambda_2 < 0$ if $c \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0, \frac{1}{\sqrt{3}}\right)$; $\lambda_1, \lambda_2 > 0$ for $|c| > 1$; and $\lambda_1 > 0, \lambda_2 < 0$ if $c \in \left(-1, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$. Thus by using Hartman-Grobman theorem, we can conclude that while $H_e > 0$ the equilibrium point C is an unstable node if $c \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0, \frac{1}{\sqrt{3}}\right)$, a stable node for $|c| > 1$, and a saddle point if $c \in \left(-1, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$. Also note that while $H_e < 0$ the equilibrium point C is a stable node if $c \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0, \frac{1}{\sqrt{3}}\right)$, an unstable node for $|c| > 1$, and a saddle point if $c \in \left(-1, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$.

Next we consider $Q = 3b^2\rho_{dm}$. In the spatially homogeneous and isotropic universe, the continuity equation and conservation equation for DE assume the form as,

$$\begin{aligned} \dot{H} &= -\frac{3H^2}{2} + \frac{\rho_{de}}{6} + \frac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH} + \frac{3b^2H}{2} - \frac{b^2\rho_{de}}{2H}, \\ \dot{\rho}_{de} &= \frac{2\rho_{de}^{\frac{3}{2}}}{\sqrt{3}c} - 2H\rho_{de}. \end{aligned} \qquad (20)$$

The above two equations forms an autonomous system. We study the stability behaviour of the system. The only physically meaningful equilibrium point corresponding to the above system is $D \equiv (b^2, 3b^4c^2)$ provided $c \neq \pm 1$. The Jacobian matrix of the system $(\ref{eq31}-\ref{eq32})$ is

$$J(H,\rho_{de}) = \begin{pmatrix} -3H - \dfrac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH^2} + \dfrac{3b^2}{2} + \dfrac{b^2\rho_{de}}{2H^2} & \dfrac{1}{6} + \dfrac{\sqrt{\rho_{de}}}{2\sqrt{3}cH} - \dfrac{b^2}{2H} \\ -2\rho_{de} & \dfrac{\sqrt{3\rho_{de}}}{c} - 2H \end{pmatrix}$$

The eigenvalues of the Jacobian matrix at the point D are

$$\lambda_{1,2} = \frac{1}{4}\left(b^2c^2 - b^2 \mp \sqrt{b^4c^4 - 26b^4c^2 + 25b^4}\right).$$

We can see that for $b = 0$ the eigenvalues are 0 but then the coordinate of the equilibrium point becomes $(0,0)$ but at this point a singularity occurs (as H presents in the denominator part of first equation of (20)). So the equilibrium point D is a hyperbolic point for all b and c. Using Hartman-Grobman theorem, we conclude that the equilibrium point D is an unstable focus if $c \in (-5,-1) \cup (1,5)$, for $|c| > 5$ the equilibrium point D behaves as an unstable node, and the type of stability of the equilibrium point D is a saddle node if $|c| < 1$.

At last we consider the interaction term $Q=\frac{3b^2\rho_{dm}\rho_{de}}{H}$. For the given term, the equations (20) takes the following form

$$\begin{aligned}\dot{H} &= -\frac{3H^2}{2}+\frac{\rho_{de}}{6}+\frac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH}+\frac{3b^2\rho_{de}}{2}-\frac{b^2\rho_{de}^2}{2H^2},\\ \dot{\rho}_{de} &= \frac{2\rho_{de}^{\frac{3}{2}}}{\sqrt{3}c}-2H\rho_{de}.\end{aligned} \tag{21}$$

The above system is an autonomous system. We observe the system using different dynamical tools. This autonomous system extracts two line of equilibrium points $E_1=(H_e,3H_e^2)$ when $c^2=1$ and $E_2=(H_e,3c^2H_e^2)$ when $3b^2c^2=1$. Jacobian matrix corresponding to the above system is

$$J(H,\rho_{de})=\begin{pmatrix}-3H-\frac{\rho_{de}^{\frac{3}{2}}}{3\sqrt{3}cH^2}+\frac{b^2\rho_{de}}{H^3} & \frac{1}{6}+\frac{3b^2}{2}+\frac{\sqrt{\rho_{de}}}{2\sqrt{3}cH}-\frac{b^2\rho_{de}}{H^2}\\ -2\rho_{de} & \frac{\sqrt{3\rho_{de}}}{c}-2H\end{pmatrix}$$

Now let's determine the stability of E_1 for $c=1$. In this case, the eigenvalues of the Jacobian matrix at the point E_1 are 0 and $3H_e(3b^2-1)$. It follows that the equilibrium point E_1 is always nonhyperbolic. Plotting the system numerically in a suitable graphing utility, we can observe that the line of equilibrium point E_1 is stable when $\left(H_e<0,|b|>\frac{1}{\sqrt{3}}\right)$ or $\left(H_e>0,|b|<\frac{1}{\sqrt{3}}\right)$ and E_1 is unstable when $\left(H_e>0,|b|>\frac{1}{\sqrt{3}}\right)$ or $\left(H_e<0,|b|<\frac{1}{\sqrt{3}}\right)$.

Now for $c=-1$, the eigenvalues of the Jacobian matrix at the point E_1 are

$$\lambda_{1,2}=\frac{H_e}{2}\left(-7+9b^2\mp\sqrt{17+90b^2+81b^4}\right).$$

We can observe that while $H_e>0$ the eigenvalue $\lambda_2<0$ for all b and $\lambda_1>0$ when $|b|>0.385$ and $\lambda_1<0$ when $|b|<0.385$. Further, we can see that while $H_e<0$ the eigenvalue $\lambda_2>0$ for all b and $\lambda_1<0$ when $|b|>0.385$ and $\lambda_1>0$ when $|b|<0.385$. Thus by using Hartman-Grobman theorem, we can conclude that for $c=-1$, the equilibrium point E_1 is a saddle node if $|b|>0.385$ and a stable node $|b|<0.385$ while $H_e>0$; and the equilibrium point E_1 is an unstable node if $|b|<0.385$ and a saddle node $|b|>0.385$ while $H_e<0$.

Now let's discuss the stability criteria of the equilibrium point E_2. The eigenvalues of the Jacobian matrix at the point E_2 are

$$\lambda_{1,2}=\frac{1-2H_e^2-c^2H_e^2}{2H_e}\mp\frac{\sqrt{1-8H_e^2-2c^2H_e^2+4H_e^4+16c^2H_e^4+c^4H_e^4}}{2H_e}.$$

For the hyperbolic case, using the Hartman-Grobman theorem, the stability features of the equilibrium point E_2 in the regions of the parameter space (H_e,c) are shown in the figure 3.

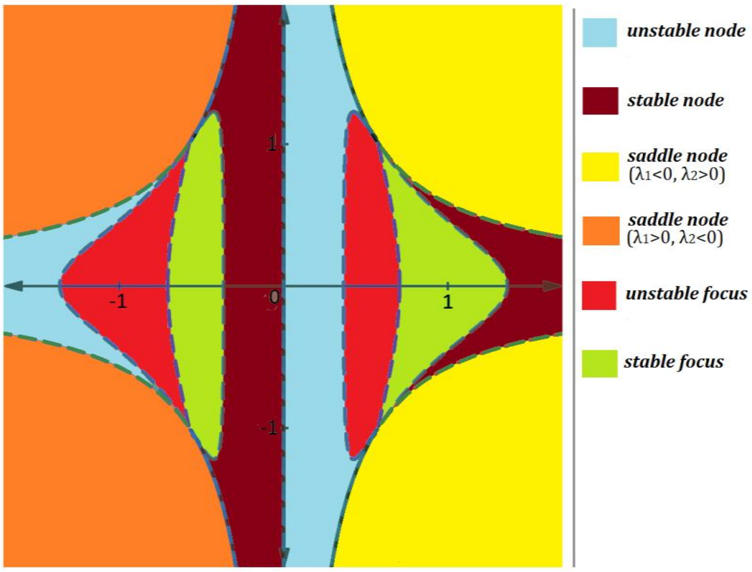

Figure 3. This figure shows the regions of the parameter space (H_e,c) corresponding to the stability features for the equilibrium point E_2. The horizontal axis represents parameter 'H_e 'and the vertical axis represents parameter 'c'.

3. MODIFIED HOLOGRAPHIC RICCI DARK ENERGY MODEL

Here we introduce infrared cutoff L with modified holographic Ricci dark energy in terms of $\dot{H}$ and H^2 as

$$\rho_{de} = \frac{2}{\alpha-\beta}\left(\dot{H} + \frac{3\alpha}{2}H^2\right) \tag{22}$$

here α and β are free constants. We have chosen α, β are reference by the author[ ] calculating the parameters inmost of the cases. Also the equation of the state parameter for DE takes the form

$$\omega_d = -(\alpha-\beta) - \frac{\alpha}{\Omega_d} - \frac{1}{\Omega_d} \tag{23}$$

We estimate deceleration parameter using field equations is given by

$$q = -\frac{\ddot{a}}{2H^2} = \frac{3\alpha}{2} - \frac{3}{2}\Omega_d(\alpha-\beta)-1 \tag{24}$$

We shall now analyze the dynamical system for two different choices of the interactions.

$A.Q = 3Hb^2\sqrt{\rho_{dm}\rho_{de}}$

The evolution equation for Ω_d can be written as

$$\dot{\Omega_d} = 3H\left[-b^2\sqrt{\Omega_d-\Omega_d^2} + (\Omega_d-1)\left((\alpha-1)-(\alpha-\beta)\Omega_d\right)\right]. \tag{25}$$

The second Friedman equation can also be written as

$$\dot{H} = -\frac{3H}{2}[\alpha-(\alpha-\beta)\Omega_d]. \tag{26}$$

The equations (25) and (26) together form an autonomous system in the (Ω_d,H) plane. The only physically relevant non-isolated equilibrium point of this system is $\left(\frac{\alpha}{\alpha-\beta},H_c\right)$ for all $H_c \in R-\{0\}$. The eigenvalues of the Jacobian matrix corresponding to this autonomous system at this equilibrium point are $\frac{3H_c}{2\alpha}(\beta-\alpha-2\alpha\beta)$ and 0. From the obtained eigenvalues, we can conclude that the equilibrium point $\left(\frac{\alpha}{\alpha-\beta},H_c\right)$ is nonhyperbolic. Plotting the system numerically in a suitable graphing utility, we can observe that the equilibrium point is stable when $\left(H_c < 0, \alpha < \frac{\beta}{1+2\beta}\right)$ or $\left(H_c > 0, \alpha > \frac{\beta}{1+2\beta}\right)$ and the equilibrium point is unstable when $\left(H_c > 0, \alpha < \frac{\beta}{1+2\beta}\right)$ or $\left(H_c < 0, \alpha > \frac{\beta}{1+2\beta}\right)$.

B. $Q = 3Hb^2\rho_{dm}$

The evolution equation for Ω_d and second Friedman equation can be written as

$$\begin{aligned}\dot{\Omega}_d &= 3H(\Omega_d - 1)\left[(\alpha-1)-(\alpha-\beta)\Omega_d + b^2\right],\\ \dot{H} &= -\frac{3H^2}{2}\left[\alpha-(\alpha-\beta)\Omega_d\right].\end{aligned} \tag{27}$$

Equations (27) together forms an autonomous system. The non-isolated equilibrium points of this system are $F_1 \equiv \left(\frac{\alpha}{\alpha-\beta}, H_c\right)$ for all $H_c \in R-\{0\}$, provided $b^2 = 1$ and $F_2 \equiv (1, H_c)$, provided $\beta = 0$. The eigenvalues of the Jacobian matrix corresponding to this autonomous system at F_1 are $-3H\beta$ and 0. So the nature of the equilibrium point F_1 is non-hyperbolic. Plotting the system numerically in a suitable graphing utility, we can observe that the equilibrium point is stable when $(H_c < 0, \beta < 0)$ or $(H_c > 0, \beta > 0)$ and the equilibrium point is unstable when $(H_c > 0, \beta < 0)$ or $(H_c < 0, \beta > 0)$. Further, the eigenvalues of the Jacobian matrix corresponding to the autonomous system (27) at F_2 are $3H(b^2-1)$ and 0. So the nature of the equilibrium point F_2 is non-hyperbolic. Plotting the system numerically in a suitable graphing utility, we can observe that the equilibrium point is stable when $(H_c < 0, |b| > 1)$ or $(H_c > 0, |b| < 1)$ and the equilibrium point is unstable when $(H_c < 0, |b| < 1)$ or $(H_c > 0, |b| > 1)$.

4. COSMOLOGICAL IMPLICATIONS OF THE equilibrium pointS

Case I: The nature of the equilibrium point depends on the parameter $'c'$. The equilibrium point will may be hyperbolic or not for $0 < c^2 < 3/7$. For this range of values of $'c'$, $\omega_{de} < -1$, i.e., the dark energy is purely of phantom nature. For $u \neq 0$, there is scaling cosmological solution corresponding to the equilibrium point while $u = 0$ corresponds to purely dark energy model. For $c^2 > 3/7$, b^2 is negative, indicating that Q is negative and hence there is flow of energy in the opposite direction. Also for $c^2 > 3/7$ the dark energy fluid will not be phantom in nature. However, for $c^2 > 3/7$ the equilibrium point corresponds to an accelerating phase of expansion while for $c^2 < 3/7$, the universe may have a decelerated era of expansion depending on some restriction on u.

Case II: The equilibrium point corresponding to this interaction describes scaling solution for non-zero u. For $u = 0$, $c = 1$ the equilibrium point describes the cosmological evolution only with cosmological constant. The dark fluid is of phantom nature corresponding to scaling solution and there is always accelerated expansion for $b^2 \leq 1$. Further, $u = 0$ is not much interesting from the observational point of view as it represents the transition epoch $q = 0$.

Case III: The cosmological analysis is similar to the previous interaction. Here, $c^2 = 1$, i.e., $b^2 = 0$ can not be chosen as $Q = 0$, i.e., the two fluids become non-interacting. Though the equilibrium point is mostly dark energy dominated with the equation of state of dark energy, $\omega_{de} < -1$ but for $b^2 > 1$ (i.e., $c^2 < 1/5$) $q = 0$, i.e., there is decelerated expansion of the universe.

Case IV: The interacting model is not much interest as q is always zero. Still, it is interesting to note that though the equilibrium point represents a scaling solution with dark energy equation of state either in the phantom domain or very close to it in the quintessence domain but still there is no longer accelerated expansion. Probably, the interaction has a great influence on the evolution suppressing the role of dark energy.

Case V: Here among the two equilibrium points E_1 is more interesting than E_2 as $q = 0$ for E_2 throughout the evolution. For the equilibrium point E_1 with $c = 1$, there is always decelerated expansion of the universe provided $b^2 > 1/3$ while accelerated expansion is possible for $b^2 < 1/3$. However, $c = -1$ is mostly dark energy dominated and there is accelerated expansion.

Case VI: For this interaction HDE model, the equilibrium point with $\beta = 0$ corresponds to ΛCDM model and $q = -1$ at all instant. Hence it does not indicate any cosmic evolution. The equilibrium point for $\beta \neq 0$ is interesting. The dark energy represents exotic fluid if $\beta < \alpha$ and $\alpha > -1/2$ or $\beta > \alpha$ and $\alpha < -1/2$, in other cases ω_d corresponds to normal fluid. However, the cosmic evolution will be in accelerated mode or in decelerated mode, depending on $\beta < 2/3$ or $\beta > 2/3$. Note that $\beta = 2/3$ and $\beta = 0$ describe the two transition epochs namely $q = 0$ and $q = -1$ respectively.

5. Brief Summary

The present work is an example to show how a cosmological model can be analyzed through dynamical system analysis without solving the coupled nonlinear Einstein Field equations. In the interacting two-fluid system, a dust fluid is chosen as the cold dark matter and for the dark energy; (i) holographic dark energy with IR cut off as event horizon is chosen (ii) modified holographic Ricci dark energy model is considered in two different sections. For the interaction it is assumed that energy is transformed from DE to DM and five possible choices for the interaction terms are considered. For hyperbolic equilibrium points Hartman-Grobman theorem has been used to study the stability of the equilibrium points while for the nonhyperbolic equilibrium points, suitable Lyapunov function has been constructed and stability analysis has been done. The profile of the global analysis in finite phase-space has been presented graphically. Finally, the regions of the parameter space corresponding to the stability features for the equilibrium points has been shown graphically.

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Conflict of interest The authors declare that they have no conflict of interest.

References

1. D. N. Spergel et al. Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology. Astrophys. J. Suppl., 170:377, 2007.
2. D. N. Spergel et al. First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters. Astrophys. J. Suppl., 148:175–194, 2003.
3. P. de Bernardis et al. A Flat universe from high resolution maps of the cosmic microwave background radiation. Nature, 404:955–959, 2000.
4. S. Perlmutter et al. Measurements of Ω and Λ from 42 High Redshift Supernovae. Astrophys. J., 517:565–586, 1999.
5. Adam G. Riess et al. Type Ia supernova discoveries at z > 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution. Astrophys. J., 607:665–687, 2004.
6. Antonio Pasqua, Surajit Chattopadhyay, and Ratbay Myrzakulov. Consequences of three modified forms of holo graphic dark energy models in bulk–brane interaction. Can. J. Phys., 96(1):112–125, 2018.
7. Christian G. Boehmer, Nyein Chan, and Ruth Lazkoz. Dynamics of dark energy models and centre manifolds. Phys. Lett. B, 714:11–17, 2012.
8. Shuxun Tian, Shuo Cao, and Zong-Hong Zhu. The Dynamics of Inhomogeneous Dark Energy. Astrophys. J., 841(1):63, 2017.
9. Stephen D. H. Hsu. Entropy bounds and dark energy. Phys. Lett. B, 594:13–16, 2004.
10. Andrew G. Cohen, David B. Kaplan, and Ann E. Nelson. Effective field theory, black holes, and the cosmological constant. Phys. Rev. Lett., 82:4971–4974, 1999.
11. Miao Li, Xiao-Dong Li, Shuang Wang, Yi Wang, and Xin Zhang. Probing interaction and spatial curvature in the holographic dark energy model. JCAP, 12:014, 2009.
12. Miao Li, Xiao-Dong Li, Shuang Wang, and Xin Zhang. Holographic dark energy models: A comparison from the latest observational data. JCAP, 06:036, 2009.
13. Miao Li. A Model of holographic dark energy. Phys. Lett. B, 603:1, 2004.
14. Nairwita Mazumder and Subenoy Chakraborty. Validity of the generalized second law of thermodynamics of the universe bounded by the event horizon in holographic dark energy model. Gen. Rel. Grav., 42:813–820, 2010.
15. Changjun Gao, Fengquan Wu, Xuelei Chen, and You-Gen Shen. A Holographic Dark Energy Model from Ricci Scalar Curvature. Phys. Rev. D, 79:043511, 2009.
16. R. J. Yang, Z. H. Zhu, and F. Wu. Spatial Ricci scalar dark energy model. Int. J. Mod. Phys. A, 26:317–329, 2011.
17. L. N. Granda and A. Oliveros. New infrared cut-off for the holographic scalar fields models of dark energy. Phys. Lett. B, 671:199–202, 2009
18. Nilanjana Mahata and Subenoy Chakraborty. A dynamical system analysis of holographic dark energy models with different IR cutoff. Mod. Phys. Lett. A, 30(27):1550134, 2015.
19. Sergio del Campo, Ramon Herrera, and Diego Pavon. Toward a solution of the coincidence problem. Phys. Rev. D, 78:021302, 2008.

20. Diego Pavon and Winfried Zimdahl. Holographic dark energy and cosmic coincidence. Phys. Lett. B, 628:206–210, 2005.
21. Bin Wang, Yun-gui Gong, and Elcio Abdalla. Transition of the dark energy equation of state in an interacting holographic dark energy model. Phys. Lett. B, 624:141–146, 2005.
22. Bin Wang, Chi-Yong Lin, and Elcio Abdalla. Constraints on the interacting holographic dark energy model. Phys. Lett. B, 637:357–361, 2006.
23. Shuang Wang, Yi Wang, and Miao Li. Holographic Dark Energy. Phys. Rept., 696:1–57, 2017.